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Topological Graphics

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Abstract

Topological graphics opens up completely new worlds in computer graphics applications. It is supported by advances in modern algebraic topology: homotopy theory and cellular spatial structures in particular. Topological graphics lays out the framework to represent digital contents in a few orders of magnitude efficient. It guides graphics software design to make it minimal and reusable. This progress report on our own frontier researches give abundant of examples as well as the brief summary of the theoretical foundation.

1. Introduction

Solving numerous important problems in our life present us marvelous opportunities to enjoy finding neat solutions. This paper shows you a part of our fun on doing that. First, we go from well know problems of characterizing mountain terrains we see when we go hiking. The problem is easily solved by applying the Morse lemma and the Reeb graph in differential topology. Another interesting problems around us is how to characterize our clothing. One aspect is basically identical to the previous problem of identifying characteristic points such as peaks, pits and passes, with some extensions to add ridges and ravines. Charactering clothing has the other interesting problems such as how to identify flexible shapes. The flexibility destroys geometry requiring more general and hence abstract characteristics that happens to be cellular spatial structure problems in algebraic topology, homotopy in particular.

There have been two major difficulties in pursuing this type of researches. One is the lack of curriculum to train computer scientists in algebraic topology. The other is finding appropriate peer reviewers stemming from the first difficulty. Mathematicians know mathematics and they say the mathematics applied are trivial and nothing new. The problem being attacked are in the domain of computer science. They do not say nothing new in mathematics but just say nothing new to reject the papers. On the other hand, computer scientists say why they care such strange notations and hard mathematics to conclude to reject the papers.

Historically all new research areas and industrial applications, now respected and valued, have come through this type of difficulties: quantum theory and semiconductors as its applications, for example. Therefore the most appropriate academic standing for us is to attack ever expanding key problems without concerning the immediate popularity. The two difficulties explained above have been solved in part by creating a new department, a new school and new university as well as new professional societies and new refereed journals. Yet mathematics curricula for computer scientists still teaches a few hundred to a few thousand years old materials such as graphs and polygons, neglecting recent and very useful mathematical findings and causing a notorious polygon explosion problem in computer graphics.

It is noted that the activities on the web trading a GNP equivalent a day require web-based graphics. Varieties of web-based graphics have been developed mainly in the Java environment for cross platform applications on the web. For this purpose, the generality of topological graphics plays a central role. Web-based graphics is in cyberspaces on the web. From here on, we look at topological graphics in this perspective [8].

2. Requirements for modeling cyberspaces

Let us derive a set of requirements for designing the cyberspaces where we install our differential topological modelers. We abstract the requirements from wide varieties of real world cases.

A. *Case 1 Topographical Applications*

Living on the earth, the human races have been measuring the topography of the earth surface for thousands of years to make topographical maps. A topographical map preparation process in a general case is through the concurrent measurement of topographical height differences over distance at various points on the topography. Each person measuring the topography assumes a local coordinate system at the place being measured. On the globe, it looks like in Fig.1. A locally correct but globally wrong model is being used even now, and we know how to generate a globally correct world map from them.

Very large numbers of area maps have been accumulated and they are in daily use in the world. These area maps are integrated to produce global maps such as European maps, Asian maps and world maps. It is well known that area maps yield sometimes a few hundred meters discrepancies in topographical height values at the boundaries of neighboring countries. We characterize the topography of each area by a set of equi-height lines that are two dimensional (2D) and, finally by a set of characteristic points such as peaks, pits and passes that are one dimensional (1D). The whole topography is a solid body that is three dimensional (3D) and their changes by eruption, erosion, vegetation and other causes are 4 dimensional (4D). It is divided into solid topographical objects such as mountains and lakes (lakes are usually pits) and their changes.

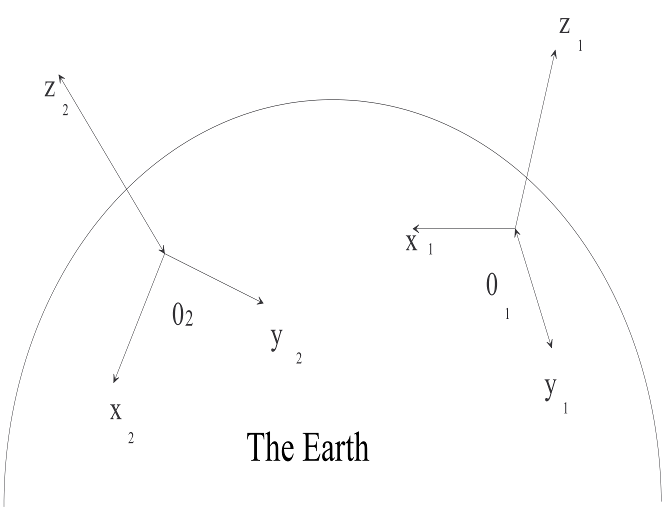


Fig. 1 Local coordinate systems at different topological measurement locations, exhibiting cellular spatial structures

B. *Case 2 Flexible Manufacturing Automation*

One of the most advanced intelligent mechatronics is realized as flexible manufacturing automation employing groups of cooperating robots to work at various locations of a manufacturing line from different directions. Each robot, then, has a local coordinate system to define its work space and configuration space. Each segment of a robot has 6 degrees of freedom both in the work space and the configuration space, 3 for locations and the other 3 for rotation angles. A work space is often called a work cell. The work cells are lined up with a certain amount of overlaps to cooperate each other to form a manufacturing line as shown in Fig. 2 .

Currently, the number of the joints of industrial roots are limited to less than ten and hence cannot perform elaborate high quality jobs as good as human beings [2]. Highly multi-joints robots require a large number of work cell. This is one of the most attractive research areas for intelligent processing systems. To achieve high quality productivity, cellular spatial structures can provide an ideal ground for designing individually defined work cells working interactively and cooperatively to realize an overall integrated production line.

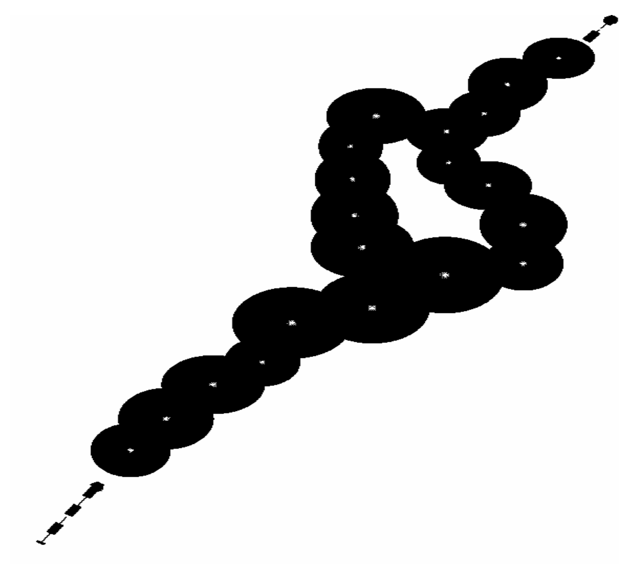


Fig. 2 A manufacturing line, each circle denoting a work cell.

C. *Case 3 Financial Applications*

Financial people, financial traders in particular, are observing the dynamic changes of the values of various financial parameters for trading purposes, such as exchange rates between different currencies in different countries and stock prices at international stock trading centers. Before the financial market reaches the peaks and pits of exchange rates and stock values, actually at the passes, financial trading people make themselves well prepared to execute transactions of selling and buying.

Financial applications, thus requires a local value system as a local coordinate system at each site in the world, concurrently working. Each site is a financial cell, including 1D financial critical points, 2D financial equi-value lines and the whole financial structures as 3D objects as well as their changes as 4D objects. In conclusion, cellular structured financial spaces are defined at individual sites to perform financial trading at individual sites, while the integration mechanisms to go from one cell to another is required to carry out the global financial trading internationally.

D. Case 4 Medical and dental applications

Ultrasonic computed tomography is among the most popular medical imagery techniques for advanced

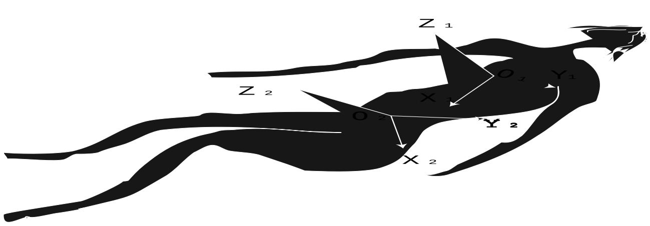


Fig. 3 Ultrasonic computed tomography inspection exhibiting cellular spatial structures

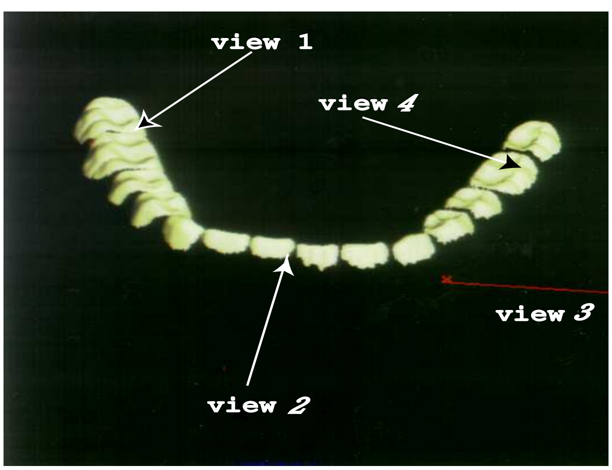


Fig. 4 Dental inspection and the inspected teeth as seen from multiple viewpoints, illustrating a case of cellular spatial structures displayed as an integrated image.

equipment to the patient body at different locations as needed. Each location has its own local coordinate system depending on the position and direction of the equipment as seen in Fig. 3.

Dental inspections similarly proceed: a dental doctor view the patient's teeth from different positions and from various directions, either visually or using an X-ray device (Fig. 4).

In both cases, doctors start inspections from problematic areas, and proceed to the neighborhood to check the spread of the disease and the symptoms to complete the diagnosis of the patient body as an integrated entity, not just as a collection of organs.

E. Case 5 Human Mental Space Modeling as Cyberspace Modeling

Any experts with quality skills perform the works in various specialization including those shown above based on the expert knowledge and experiences accumulated in their memory as the cellular structured spaces. In this sense, human mental modeling is equivalent to a set of requirements for cyberspaces abstracted from the typical cases of real world applications as explained so far.

In our human mind, we remember events and occasions as scenes, usually loosely related and coming into our mind in parallel. This means that the human memory is configured as a cellular structured space where cells are partially intersecting with each other. Each cell contains the 1D, 2D, 3D and 4D objects in the scene and also the characteristics of the scene such as singularities of the objects and object changes including their movements. The singularities serve as the indexes of the objects and their changes.

3. Architecture design of cyberspaces

By architecturally designing and implementing the human mental modeling in the cyberspaces of networked computers, we can establish the sound spaces where we can place quality work, quality products and quality services as the results of automating quality generation through intelligent processing systems towards the 3rd industrial revolution. For the necessary automation to be able to carried out correctly without ambiguity while maintaining the *generality to cover wide ranges of application cases*, we have to carefully design the cellular spatial structure architecture.

A. Set theoretical design

First of all, we start our design work from defining a collection of objects we are looking at. To be able to conduct automation on such a collections by using computers as intelligent machines, each collection has to be a *set* because computers are built as set theoretical machines. Intuitively, a *set* X is a collection of all objects x having an identical property, say P(x). Symbolically $X = \{x : P(x)\}$. Any object in a set is called an element. A set without an element is named the empty set $\varnothing$. A set is said *open* if all of its elements are interior. Given sets X and Y, computers perform set theoretical operations such as the union $X \cup Y$, the intersection $X \cap Y$ and the difference $X - Y$ (also denoted as $x \setminus y$). Suppose we begin our cyberspace architecture design from a set X as the initial cyberspace. Given all elements u of an unknown cyberspace U, if they are confirmed to be the elements of our cyberspace X, the unknown cyberspace is called a *subset* or a subcyberspace and denoted as $U \subseteq X$. Thus, the subset check is automatically performed by processing $(\forall u)(u \in U \rightarrow u \in X)$. The *closure* $\bar{U}$ of U is the intersection of all closed subsets of X, containing U. In other words, the closure $\bar{U}$ is the elements of X that are not the exterior elements of U. The set of all the subsets of X, $\{U : U \subseteq X\}$, is called a *power set* 2^{x}. The power set is quite useful to design the cyberspace as consisting of sub-cyberspaces.

B. Topological design

Now, we go into the business of designing the cyberspace as the union of the sub-cyberspaces and their overlaps. The cyberspace thus designed is generally called a *topological space* (X, T) where $T \subset 2^{x}$. Designing a topological space is automated by the following specification:

1) $X \in T$ and $\varnothing \in T$;
2) For an arbitrary index set J,
$\forall j \in J\,(U_j \in T) \rightarrow \cup_{j \in J} U_j \in T$;
3) $U, V \in T \rightarrow U \cap V \in T$.

T is said to be the *topology* of the topological space (X, T). Given two topologies T_1 and T_2 on X such that $T_1 \subset T_2$, we say T_1 is *weaker* than T_2 (or T_2 is *stronger* than T_1). For simplicity, we often use X instead of (X, T) to represent a topological space whenever no ambiguity arises. When we see two topological spaces (X, T) and (Y, T'), how we can tell (X, T) and (Y, T') are equivalent ? Here is a criteria for us to use computers to automatically validate that they are topologically equivalent. Two topological spaces (X, T) and (Y, T') are *topologically equivalent* (or *homeomorphic*) if there is a function $f : (X, T) \rightarrow (Y, T')$ that is continuous, and its inverse exists and is continuous. We write $(X, T) \cong (Y, T')$ for (X, T) to be homeomorphic to (Y, T'). Then, how to validate the *continuity* of a function ? It amounts to check $\forall B \in T'$, $f^{\leftarrow} B \in T$, where $f^{\leftarrow} B$ means the inverse image of B by f. $= f(a) = g(a)$). F is said *homotopic* to g relative to A, and denoted as $f \simeq g$ (rel A). Now here is a new design problem. That is, how we can design two topological spaces X and Y to be *homotopically equivalent* $X \simeq Y$, namely *of the same homotopy type*. It is done by designing $f : X \rightarrow Y$ and $h : Y \rightarrow X$ such that $h \circ f \simeq 1_X$ and $f \circ h \simeq 1_Y$, where 1_X and 1_Y are identity maps $1_X : X \rightarrow X$ and $1_Y : Y \rightarrow Y$.

Homotopy equivalence is more general than topology equivalence so that homotopy equivalence can identify a changing cyberspace that is topologically not any more equivalent after the change. As an example, let us look at the human body. By the way, the human body is so complex that it is often called a micro cosmos, and it itself is a cyberspace. What we are looking at is a very simple case for an illustration purpose only, but is enough for our purpose. Given a silhouette of the human body of a person dancing. Fig. 5 shows how we can identify it automatically while the silhouette shape is changing during the dance. In manufacturing automation, while a component goes through a manufacturing line, for example to get deformed to a desired shape by a group of automated press machines, the deformation process is specified by a homotopy and validated by homotopy equivalence. In medical imagery, from a set of computed tomographical 2D sliced images, we can reconstruct 3D images homotopically in a following way: considering that a slice is homotopically deformed to the next slice, and then from the algorithm derived from the

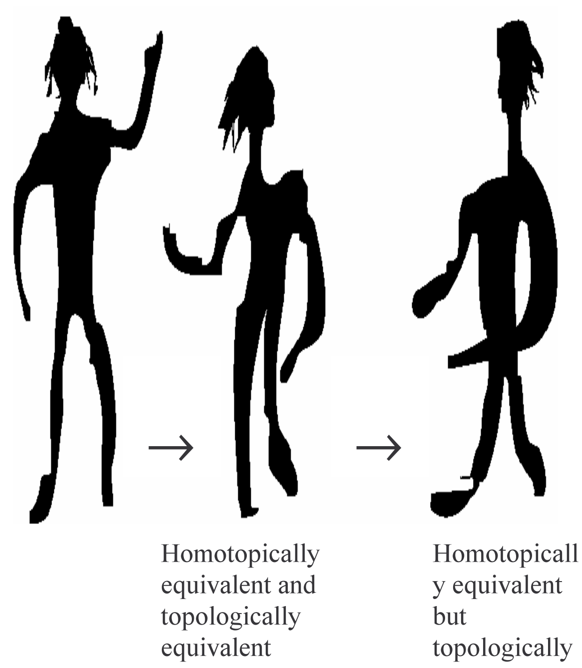


Fig. 5 Homotopy equivalence is more general than homeomorphism as topological equivalence.

law of the formation of the organ of which the images are obtained tomographically we can record all the intermediate shapes between the two neighboring slices faithfully without loss. The same applies to the topography case. Further applications are in financial trading areas where from discrete financial data, we can build the entire features automatically and then conduct automated financial trading in a cyberspace based on the expected financial value change shapes thus homotopically derived.

D. Cellular structured space design

Now we are finally at the stage of designing *cellular structured spaces*, in short a *cellular space* [7]. First of all, a *cell* is designed as a topological space X that is topologically equivalent (namely homeomorphic) to an arbitrary dimensional (say n-dimensional) ball $\mathcal{B}^n$, and called an n-cell. From X, we can design a finite or infinite sequence X_p of cells that are subspaces of X, indexed by integer $\mathbb{Z}$, namely $\{ X_p : p \in \mathbb{Z} \}$ called a *filtration*, such that

X_p covers X (or, X_p is a covering of X),
namely,
$X = \cup_{p \in \mathbb{Z}} X_p$, and
X_{p-1} is a subspace of X_p,
namely,
$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_{p-1} \subseteq X_p \subseteq \ldots \subseteq X$.

We can design, thus, a quite general cellular space called a *filtration space* for a cellular space X, as a space X with a filtration designed above, and denote it by $\{ X; X_p : p \in \mathbb{Z} \}$. We can actually build a little bit more structured cellular space, and hence not as general as a filtration space. It is called a closure finite and weak topology space, abbreviated as a *CW – space* constructed by the closed subspaces X_p of X. It is enough for our design with finite numbers of cells. If we need to think about an infinite case, some extra care is required. Further, as in the most cases in natural sciences as seen in theoretical physics, smoothness, namely the existence of continuous derivatives of all orders, is assumed, and sometimes *diffeomorphism,* namely differentiability with a differential inverse is further assumed to turn a CW-complex into a more special case named a *manifold*. However, in designing cyberspaces that are quite general, usually we cannot have the luxury of enjoying such a limited space built on top of such assumptions.

E. Manifold design

As a typical application of manifolds we take an area guide map generation. A guide map is a sheet of paper where scenes are pasted together as seen from the vista points appropriate for individual sceneries [7], [8].

Figure 6 gives a case of a guide of a part of this famous scenic area, Hakone, pasting the two views.

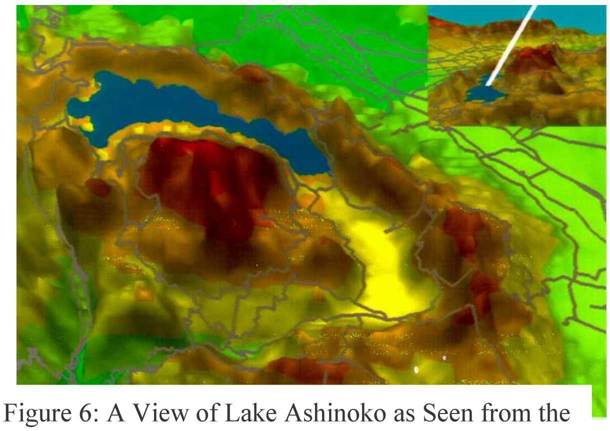


Figure 6: A View of Lake Ashinoko as Seen from the Top of Mt. Kamiyama.

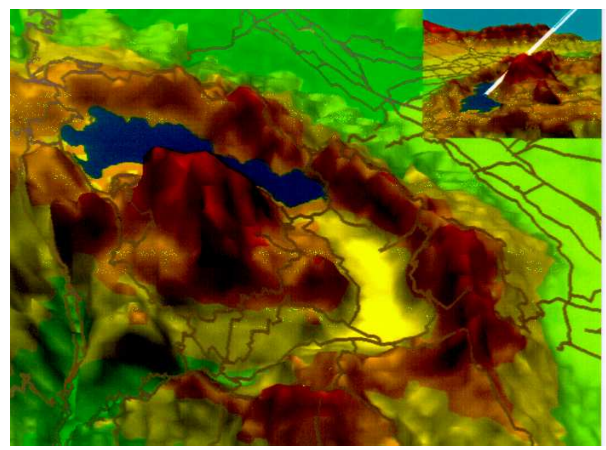


Figure 7: A View of Mt. Kamiyama as Seen from the Opposite Side of Lake Ashinoko.

Figure 6 is a view of Lake Ashinoko as seen from the top of Mt. Kamiyama and Figure 7 is another view, the view of Mt. Kamiyama as seen from the opposite side of Lake Ashinoko. Figure 8 is a case of a guide map obtained pasting the two views and the views of the neighboring. The cellular overlaps for gluing are normalized by the *partition of unity*.

We have been studying further applications of cellular structured spaces. However, in designing cyberspaces that are quite general, usually we cannot have the luxury of enjoying such a limited space built on top of such assumptions. The most of the real world applications have singularities, and non-diffeomorphic cellular spaces. Hence, we usually use filtration spaces to model both virtual and real worlds in cyberspaces.

The research in cellular structured spaces in technological applications has to go through careful case

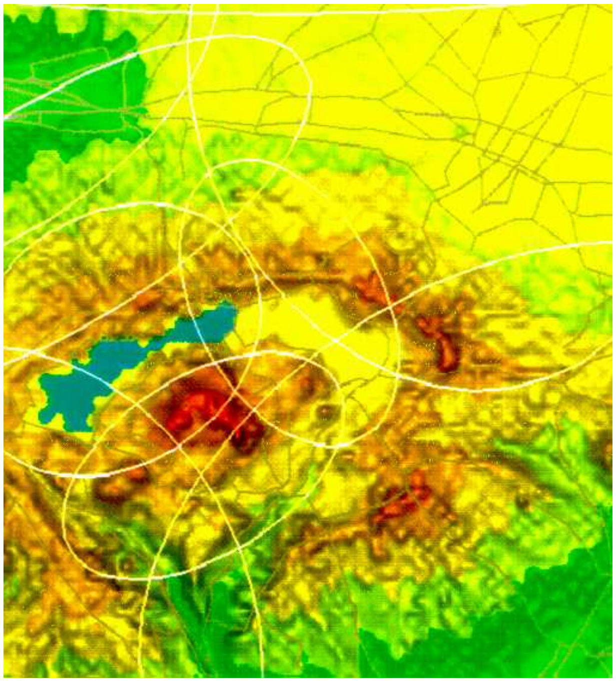


Figure 8: An Example of a Guide Map of Lake Ashinoko and Mt. Kamiyama Area in the Hakone Scenic Sites.

studies and investigations that will also require significant amount of basic research in mathematics. There is a significant problem in developing curriculums and courseware to educate and train students and technological professionals in all application domains.

F. *Differential topological design: Morse theoretical model and Reeb graph model*

Definition A critical point x of f is called *nondegenerate* if d^2f is *non*degenerate at that point. This is equivalent to the condition det $d^2f \neq 0$ at x. The *index* of x is the index of d^2f at x. The *nullity* of x is the nullity of d^2f at x.

These definitions do not depend on the choice of a local coordinate system. In this paper we will deal mostly with nondegenerate critical points.

Definition A smooth function on a smooth manifold is called a *Morse function* if all its critical points are nondegenerate.

It can be proved using Sard's theorem that Morse functions exist on any smooth manifold. In fact, any smooth function on a smooth manifold can be approximated as closely as desired by a Morse function. Nondegenerate critical points are *isolated* (that is, there cannot be a sequence of nondegenerate critical points converging to a nondegenerate critical point); in particular, a Morse function on a compact manifold has only finitely many critical points, and they are isolated.

The fact that nondegenerate critical points are isolated follows from this result. which is proved in [11], for example:

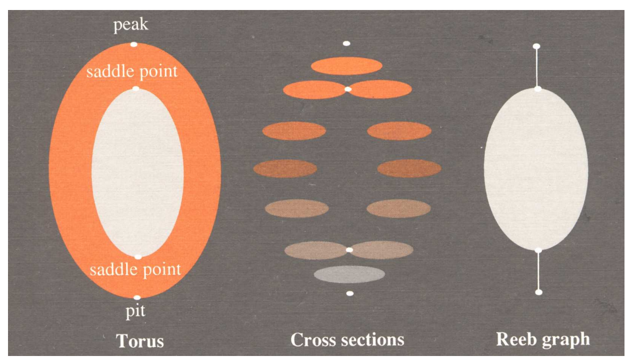
Lemma (Morse's Lemma). *If x_0 is a nondegenerate critical point of a function f on a manifold M, there is some open neighborhood of x_0* in M *and a set of local coordinates x^1, ...,x^n such that, in these coordinates, f has the form*

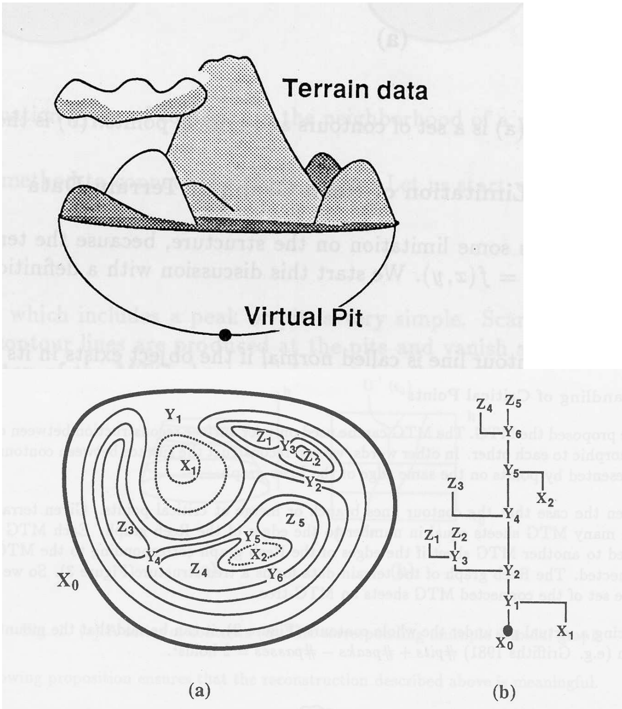
$$f(x) = f(x_0) - (x^1)^2 - \ldots - (x^\lambda)^2 + (x^{\lambda+i})^2 + \ldots + (x^n)^2,$$

where λ is the index of the critical point.

Thus, it is always possible to choose local coordinates in the neighborhood of a nondegenerate critical point so that the function in this neighborhood is a diagonalized quadratic function when expressed in these coordinates. Note that we are dealing here with an exact equality: there are no additional higher-order terms.

The Morse lemma and the Reeb graph are powerful tools to abstract the characteristics of 3D shapes. The figures below show some examples. Kergosien [12] has been pioneering researches in this area including medical applications.





4. Conclusions

In dealing with topological graphics, you may already have noticed that a cellular spatial structure including CW-comples is an extension of the concept of a *poset* (a partially ordered set); a poset is itself an extension of the concept of a *graph*. In computer science, set theory as seen in the Z system for software design specification, graph theory in the entity-relationship data models, and posets in communication network design are so far among the advanced approaches.

For web-based graphics, because of their extreme complexity, we need to use far more general modeling approach than those widely used so far such as cellular spatial structures. People can work locally within each cell instead of getting their views scattered by the global world; a cellular modeling based on cellular spatial structures provide enough assurance to get the cells put together for global integration in a well designed manner. On my homepage http://www.kunii.tv I have posted two applications of cellular modeling [13, 14].

The research reported here is still at an early stage after over a decade of intensive work. The area is wide open to anybody interested in the global world on the web.

5. References

[1] A. T. Fomenko, *Differential Topology and Geometry*, Plenum Publishing, New York; 1987.

[2] T. L. Kunii, Y. Tsuchida, Y. Arai, H. Matsuda, M. Shirahama, S. Miura, "A model of hands and arms based on manifold mappings", *in Communicating with Virtual Worlds*, N. M. Thalmann and D. Thalmann (eds.), Springer-Verlag, Tokyo and Heidelberg; 1993, pp. 381-398.

[3] A. T. Fomenko and T. L. Kunii, *Topological Modeling for Visualization*, Springer-Verlag, Tokyo and Heidelberg; 1997.

[4] T. L. Kunii, "Cognitive technology and differential topology: The importance of shape features", *in Cognitive Technology in Search of Humane Interface*, B. Gorayska and J. Mey (eds.), North-Holland, Amsterdam; 1996, pp. 337-345.

[5] T. L. Kunii, T. Noma and K. J. Lee, "SYDEM: A new approach to computer-aided design of assemblies and assemblability testing", *in Visual computing: Integrating Computer Graphics with Computer Vision*, T. L. Kunii (ed.), Springer-Verlag, Tokyo and Heidelberg,; 1992, pp. 469-479.

[6] Y. Shinagawa and T. L. Kunii, "The homotopy model: A generalized model for smooth surface generation from cross sectional data", *The Visual Computer Journal*, vol. 7, no. 2-3, 1991, pp. 72-86.

[7] F. Fritsch and R. A. Piccinini, *Cellular Structures in Topology*, Cambridge University Press, Cambridge; 19990.

[8] T. L. Kunii, "The 3rd Industrial Revolution through Integrated Intelligent Processing Systems", Proceedings of IEEE First International Conference on Intelligent Processing Systems, October 28-31, 1997, Beijing, China (ICIPS*'97*), pp. 1-6, The Institute of Electrical and Electronics Engineers, New York, NY, U.S. A.

[9] T. L. Kunii and S. Takahashi, Area Guide Map Modeling by Manifolds and CW-Complexes, In: Modeling in Computer Graphics (Proceedings of IFIP TC5/WG5.10 Second Working Conference on Modeling in Computer Graphics, June 28 - July 2, 1993, Genoa, Italy), B. Falcidieno and T. L. Kunii (eds.), pp. 5-20, Springer-Verlag, Berlin Heidelberg, 1993.

[10] S. Takahashi and T. L Kunii, Manifold-based multiple-viewpoint CAD: A Case Study of Mountain Guide-Map Generation, Computer Aided-Design Journal, Vol.26, No.8, August 1994, pp.622-631, Butterworth-Heinemann.

[11] V. Guillemin and A. Pollack, "Differential Topology'', pp. 421974, Prentice-Hall, 1974.

[12] Y. L. Kergosien, “Topology and Visualization: From Generic Singularities and Combinatorial Shape Modeling” in Modern Geometric Computing for Visualization, T. L. Kunii and Y. Shinagawa (editors), pp. 31-54, Springer, 1992.
[13] T. L. Kunii, “Valid computational shape modeling: design and implementation”, International Journal of Shape Modeling, World Scientific, Vol. 5, No. 2, pp. 123-133,December 1999.
[14] T. L. Kunii and H. S. Kunii, “A Cellular Model for Information Systems on the Web - Integrating Local and Global Information-”, 1999 International Symposium on Database Applications in Non-Traditional Environments (DANTE'99), November 28-30,1999, Heian Shrine, Kyoto, Japan, Organized by Research Project on Advanced Databases, in cooperation with Information Processing Society of Japan, ACM Japan, ACM SIGMOD Japan, pp. 19-24, IEEE Computer Society Press, Los Alamitos, California, U. S. A.