Algebraic Modeling for Cyberworld Design*

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0. Prologue: What are cyberworlds?

Cyberworlds are being formed on the web either intentionally or spontaneously, with or without design. Widespread and intensive local activities are melting each other on the web globally to create cyberworlds. What is called e-business including electronic financing has been conducted in cyberworlds and has gone beyond a national finance level in its scale. Without proper modeling, cyberworlds have continued to grow chaotic and are now out of human understanding and control. My discovery of cyberworlds goes back to [5].

A novel information model we named "an adjunction space model" serves to globally integrate local models. As an information model, it is also applicable to the category of irregular data models that capture spatio-temporal aspects of information worlds. Mathematically it is based on an incrementally modular abstraction hierarchy including cellular spatial structures in a homotopy theoretical framework [1, 2].

1. Set theoretical design

First of all, we start our design work of cyberworlds from defining a collection of objects we are looking at to construct them in cyberspaces. To be able to conduct automation on such collections by using computers as intelligent machines, each collection has to be a *set* because computers are built as set theoretical machines. Intuitively, a *set* X is a collection of all objects x having an identical *property*, say P(x). Symbolically $X = \{x \mid P(x)\}$. Any object in a set is called an element. A set without an element is named the *empty set* ϕ. A set is said *open* if all of its elements are interior. Given sets X and Y, computers perform s*et theoretical operations* such as the union $X \cup Y$, the intersection $X \cap Y$, the difference $X - Y$ (also denoted as $x \setminus y$), and the negation $\neg X$. Suppose we begin our cyberspace architecture design from a set X as the initial cyberspace. Given all elements u of an unknown cyberspace U, if they are confirmed to be the elements of our cyberspace X, the unknown cyberspace is called a *subset* of X or a subcyberspace of X and denoted as $U \subseteq X$. Thus, the subset check is automatically performed by processing $(\forall u)\ (u \in U \rightarrow u \in X)$. The *closure* $\bar{U}$ of U is the intersection of all closed subsets of X, containing U. In other words, the closure $\bar{U}$is the elements of X that are not the exterior elements of U. The set of all the subsets of X, $\{U \mid U \subseteq X\}$, is called a *power set* of X and denoted as 2^x. It is also called the *discrete topology* of X. The discrete topology is quite useful to design the cyberspace as consisting of subcyberspaces.

2. Topological design

Now, we go into the business of designing the cyberspace as the union of the subcyberspaces of X and their overlaps. The cyberspace thus designed is generally called a *topological space* (X, T) where $T \subseteq 2^x$. Designing a topological space is automated by the following specification:

1) $X \in T$ and $\varnothing \in T$;
2) For an arbitrary index set J,
$\forall j \in J\ (U_j \in T) \rightarrow \bigcup_{j \in J} U_j \in T$;
3) $U, V \in T \rightarrow U \cap V \in T$.

T is said to be the *topology* of the topological space (X, T). Given two topologies T_1 and T_2 on X such that $T_1 \subset T_2$, we say T_1 is *weaker* or *smaller* than T_2 (alternatively, we say that T_2 is *stronger* or *larger* than

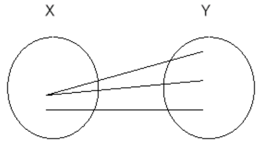
T_1. We also say T_2 is *finer* than T_1, or T_1 is *coarser* than T_2.). Obviously the *strongest topology* is the discrete topology (the power set) and the *weakest topology* is $\varnothing$. For simplicity, we often use X instead of (X, T) to represent a topological space whenever no ambiguity arises. When we see two topological spaces (X, T) and (Y, T'), how we can tell (X, T) and (Y, T') are equivalent? Here is a criterion for us to use computers to automatically validate that they are topologically equivalent. Two topological spaces (X, T) and (Y, T') are *topologically equivalent* (or *homeomorphic*) if there is a function $f : (X, T) \rightarrow (Y, T')$ that is continuous, and its inverse exists and is continuous. We write $(X, T) \cong (Y, T')$ for (X, T) to be homeomorphic to (Y, T'). Then, how to validate the *continuity* of a function f? It amounts to check, first, $\forall B \in T'$, $f^{\leftarrow} B \in T$, where $f^{\leftarrow} B$ means the inverse image of B by f, then, next, check the following:

B is open $\Leftrightarrow$ $f^{-1}(B)$ is also open in X.

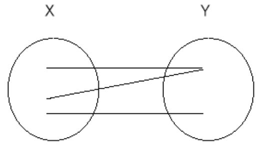
3. Functions

Given a function $f: X \rightarrow Y$, there are a total function and a partial function. For $f: X \rightarrow Y$ iff $\forall x \in X, \exists f(x)$, f is called a *total function*. A function $f: X' \rightarrow Y \mid X' \supseteq X$ is called a *partial function*, and not necessarily f(x) exists for every $x \in X$. For total functions, there are three basic types of relationships or mappings:

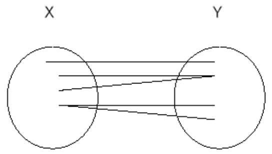
1. *Injective* or *into*, meaning $\forall x, y \in X\ x \neq y \Rightarrow f(x) \neq f(y)$; alternatively, $\forall x, y \in X\ f(x) = f(y) \Rightarrow x = y$;
2. *surjective* or *onto*, meaning $(\forall y \in Y) (\exists x \in X) [f(x) = y]$;
3. *bijective*, meaning injective and surjective.



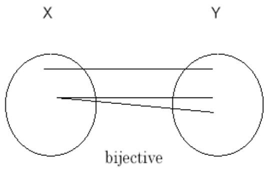
injective



not injective



surjective



bijective

4. Equivalence relations

For a binary relation $R \subseteq X \times X$ on a set X, R is :

reflexive if $(\forall x \in X) [xRx]$: reflexivity,
symmetric if $(\forall x, y \in X) [xRy \Rightarrow yRx]$:symmetry, and
transitive if $(\forall x, y, z \in X) [[xRy \Rightarrow yRz] \Rightarrow xRz]$:transitivity.

R is called an *equivalence relation* (in a notation ~) if R is reflexive, symmetric and transitive.

Given $x \in X$, a subset of X defined by $x / \sim = \{y \in X: x \sim y\}$ is called the *equivalence class* of x. Here a class actually means a set; it is a tradition, and hard to be changed at this stage. The set of all the equivalence classes X/~ is called the *quotient space* or the *identification space* of X.

$X / \sim = \{x / \sim \in 2^X \mid x \in X\} \subseteq 2^X$.

From the transitivity, for each $x \in X$, $x /\sim \neq \phi$, the followings hold:

$x \sim y \Leftrightarrow x /\sim = y /\sim$, and
$x \nsim y \Leftrightarrow x /\sim \cap y /\sim = \phi$.

This means a set X is *partitioned* (also called *decomposed*) into non-empty and *disjoint* equivalence classes.

Let us look at simple examples. In Euclidean geometry, given a set of figures, a *congruence relation* divides them into a disjoint union of the subsets of congruent figures as a quotient space; a *similarity relation* divides them into a disjoint union of the subsets of similar figures as a quotient space. Congruence and similarity relations are cases of affine transformations. A symmetry relation in group theory divides a set of figures into a disjoint union of the subsets of symmetric figures as a quotient space. In e-commerce, to be e-merchandise is an equivalence relation while e-trading is a poset (partially ordered) relation. In e-trading, a seller-buyer relation is asymmetric while an e-merchandise relation is symmetric because e-merchandise for sellers is also e-merchandise for buyers.

5. A quotient space (an identification space)

Let X be a topological space. Let f be a *surjective* (onto) and *continuous* mapping called a *quotient map* (often also called an *identification map*) that maps each point $x \in X$ to a subset (an equivalence class $x /\sim \in X /\sim$) containing x

$f: X \rightarrow X /\sim$.

Here, as explained before, "a map $f: X \rightarrow Y$ is surjective (onto)" means

$(\forall y \in Y) (\exists x \in X) [f(x) = y]$.

Suppose we take a surjective map f such that for subset X^0 of X, $X^0 \subseteq X$,

X^0 is open $\Leftrightarrow f^{-1}(X^0) \mid y \in A$ is open in X (this means f is continuous), X /~ is called a *quotient space* (or an *identification space*) by a *quotient map* (or an *identification map*) f. There is a reason why a *quotient space* is also called an *identification space*. It is because, as stated before, a quotient space is obtained by identifying each element (an equivalence class)
$x/{\sim} \in X/{\sim}$
with a point $x \in X$ that is contained in $x/{\sim}$.

6. An attaching space (an adjunction space, or an adjoining space)

Let us start with a topological space X and attach another topological space Y to it. Then,
$Y_f = Y \sqcup_f X = Y \sqcup X / \sim$
is an *attaching space* (an *adjunction space*, or an *adjoining space*) obtained by attaching (gluing, adjuncting, or adjoining) Y to X by an *attaching map* (an *adjunction map*, or an *adjoining map*) f (or by identifying each point $y \in Y_0 \mid Y_0 \subset Y$ with its image $f(y) \in X$ by a *continuous* map f) [3]. $\sqcup$ denotes a disjoint union (another name is an "exclusive or") and often a + symbol is used instead.

Attaching map f is a *continuous* map such that
$f: Y_0 \rightarrow X$,
where $Y_0 \subset Y$. Thus, the attaching space $Y_f = Y \sqcup X / \sim$ is a case of *quotient spaces*
$Y \sqcup X / \sim \; = Y \sqcup_f X = Y \sqcup X / (x \sim f(y) \mid \forall y \in Y_0)$.
The *identification map* g in this case is
$g: Y \sqcup X \rightarrow Y \sqcup_f X = Y_f = Y \sqcup X /{\sim} = (Y \sqcup X - Y_0) \sqcup Y_0$.

7. Restriction and inclusion

For any function
$g: Y \rightarrow Z$
the *restriction* of g to X ($X \subseteq Y$) is:
$g|X = g \circ i: X \rightarrow Z$
where
$i: X \rightarrow Y$
is an *inclusion*, i.e.
$\forall x \in X, i(x) = x$.

8. Extensions and retractions of continuous maps

For topological spaces X and Y, and a subspace $A \subset X$, a continuous map $f: X \rightarrow Y$ such that $f|A: A \rightarrow Y$
is called a *continuous extension* (or simply an *extension*) of a map f|A from A onto X. An extension is, thus, a partial function.

A *restriction* r is a continuous extension of an identity map $1_A: A \rightarrow A$ onto X such that
$r: X \rightarrow A$.
Then, $r|A = 1_A$.

For $A \subset X$, A is called a *deformation retract* of X, denoted by X A, if there is a *retraction*
$r: X \rightarrow A$ such that $i \circ r \sim 1_X$.

If A is a single point $A = \{a\} \subset X$, A is called *retractable* and denoted by X $*$.

9. Homotopy

Homotopy is a case of extensions. Let X and Y be topological spaces, f, g: $X \rightarrow Y$ be continuous maps, and I = [1, 0]. *Homotopy* is defined
$H: X \times I \rightarrow Y$
where for $t \in I$
H = f when t=0, and
H = g when t=1.
Homotopy is an extension of continuous maps
$H|X \times \{0\} = fi_0$, and
$H|X \times \{1\} = gi_1$
where
$i_0 = X \times \{0\} \rightarrow X$, and
$i_1 = X \times \{1\} \rightarrow X$.

Topological spaces X and Y are *homotopically equivalent* $X \simeq Y$, namely *of the same homotopy type*, if the following condition meets:
For two functions f and h
$f: X \rightarrow Y$ and $h: Y \rightarrow X$,
$h \circ f \simeq 1_X$ and $f \circ h \simeq 1_Y$,
where 1_X and 1_Y are identity maps
$1_X: X \rightarrow X$ and $1_Y: Y \rightarrow Y$.

Homotopy equivalence is more general than topology equivalence. Homotopy equivalence can identify a shape change that is topologically not any more equivalent after the change. While a shape element goes through deformation processes, the deformation processes are specified by a homotopy and validated by homotopy equivalence. As a matter of fact, from the viewpoint of the abstractness of invariance, homotopy equivalence is more abstract than set theoretical equivalence because, when we change a given set by adding or deleting elements, we can make the set homotopy equivalent by preserving the operation of add or delete and also the added or deleted elements.

10. Cellular structured spaces (cellular spaces)

A *cell* is a topological space X that is topologically equivalent (homeomorphic) to an arbitrary dimensional (say n-dimensional where n is a natural number) closed ball $\mathcal{B}^n$ called a closed *n-cell*. An open n-cell is denoted as Int $\mathcal{B}^n$ (also as $\mathring{B}^n$ and more often as e^n). $\mathcal{B}^n$ is

$\mathcal{B}^n = \{x \in \mathbb{R}^n, \|x\| \leq 1\}$,

namely a closed n-dimensional ball, and $\mathbb{R}^n$ is an n-dimensional real number.

Int $\mathcal{B}^n = \mathring{B}^n = \{x \in \mathbb{R}^n, \|x\| < 1\}$

is an open n-dimensional ball and is an *interior* of $\mathcal{B}^n$).

$\partial\mathcal{B}^n = \mathcal{B}^n - \mathring{B}^n = S^{n-1}$

is the *boundary* of $\mathcal{B}^n$, and it is an (n-1)-dimensional *sphere* S^{n-1}.

For a topological space X, a *characteristic map* $\mathcal{F}$ is a continuous function.

$\mathcal{F}: \mathcal{B}^n \to X$,

such that it is a homeomorphism:

$\mathcal{F}: \mathring{B}^n \to \mathcal{F}(\mathring{B}^n)$, and

$\mathcal{F}(\partial\mathcal{B}^n) = \mathcal{F}(\mathcal{B}^n) - \mathcal{F}(\mathring{B}^n)$.

$e^n = \mathcal{F}(\mathring{B}^n)$ is an *open n-cell*, and $\bar{e}^n = \mathcal{F}(\mathcal{B}^n)$ is a *closed n-cell.*

From a topological space X, we can compose a finite or infinite sequence of cells X^p that are subspaces of X, indexed by integer $\mathbb{Z}$, namely $\{ X^p \mid X^p \subseteq X, p \in \mathbb{Z}\}$ called a filtration, such that

X^p covers X (or X^p is a covering of X), namely,

$X = \bigcup_{p \in \mathbb{Z}} X^p$,

and X^p is a subspace of X, namely,

$X^0 \subseteq X^1 \subseteq X^2 \subseteq \ldots \subseteq X^{p-1} \subseteq X^p \subseteq \ldots \subseteq X$.

(this is called a *skeleton*). The skeleton with a dimension at most p is called a *p-skeleton*.

We also say that $C = \{X^p \mid X^p \subseteq X, p \in \mathbb{Z}\}$ is a *cell decomposition* of a topological space X, or a *partition* of a topological space X into subspaces X^p which are closed cells. (X, C) is called a *CW-complex*.

When we perform cell decomposition, by preserving cell attachment maps, we can turn cellular spaces into reusable resources. We name such preserved and shared information a *cellular database* and a system to manage it a *cellular database management system* (*cellular DBMS*).

To be more precise, according to J. H. C. Whitehead [5], given a topological space X, we *inductively* compose a *filtration* X^p with a *skeleton*

$X^0 \subseteq X^1 \subseteq X^2 \subseteq \ldots \subseteq X^{p-1} \subseteq X^p \subseteq \ldots \subseteq X$

as a topological space as follows:

(1) $X^0 \subset X$ is a subspace whose elements are 0-cells of X.

(2) X^p is composed from X^{p-1} by *attaching* (*adjunctioning*, *adjoining*, or *gluing*) to it a disjoint union $\sqcup_i\mathcal{B}_i^p$ of closed p-dimensional balls via a surjective and continuous mapping called an attaching map (an adjunction map, an adjoining map, or a gluing map)

. $F: \sqcup_i\partial\mathcal{B}_i^{\,p} \to X^{p-1}$.

In other words, we compose X^p from X^{p-1} by taking a disjoint union $X^{p-1} \sqcup (\sqcup_i\mathcal{B}_i^{\,p})$ and by identifying each point x in $\partial\mathcal{B}_i^{\,p}$, $x \in \partial\mathcal{B}_i^{\,p}$, with its image $\mathcal{F}(x)$ by a continuous mapping

$F_i = F \mid \partial\mathcal{B}_i^{\,p} : \partial\mathcal{B}_i^{\,p} \to X^{p-1}$

such that $x \sim f_i(x)$ for each index i. Thus, X^p is a quotient space (the identification space)

$X^p = X^{p-1} \sqcup (\sqcup_i\mathcal{B}_i^{\,p}) \mid (x \sim F_i(x) \mid \forall\, x \in \partial\mathcal{B}_i^{\,p})$

$= X^{p-1} \sqcup_F (\sqcup_i\mathcal{B}_i^{\,p})$

and is a case of *attaching spaces* (*adjunction spaces* or *adjoining spaces*). The map F_i is a case of *attaching maps* (*adjunction maps*, *adjoining maps* or *gluing maps*) of a cell $\mathcal{B}_i^{\,p}$. A *filtration space* is a space homotopically equivalent to a filtration. The topological space X with the skeleton $X^0 \subseteq X^1 \subseteq X^2 \subseteq \ldots \subseteq X^{p-1} \subseteq X^p \subseteq \ldots \subseteq X$ is called a *CW-space*. As a cell complex, it is called a *CW-complex* as explained before.

We thus obtain a map $\mathcal{F}$ as a case of *identification maps*

$\mathcal{F}: X^{p-1} \sqcup (\sqcup_i\mathcal{B}_i^p) \to X^{p-1} \sqcup_F (\sqcup_i\mathcal{B}_i^{\,p}) = X^p$.

A characteristic map $\mathcal{F}$ for each n-cell $\mathcal{B}_i^{\,p} = \mathcal{F}(\mathcal{B}_i^{\,p}) \in X^p$ is

$\mathcal{F}_i = \mathcal{F} \mid \partial\mathcal{B}_i^{\,p} : \partial\mathcal{B}_i^p \to X^{p-1}$.

The embedding of X^{p-1} as a closed subspace of X^p is

$\mathcal{F} \mid X^{p-1} = X^{p-1} \to X^p$.

If a CW-space is diffeomorphic, it is equivalent to a *manifold* space.

11. Suggested readings

There is no book on cyberworld design *per se* yet. To make anything computable, there is algebra. To understand basic algebra, your will find the following useful:

> "Proofs and Fundamentals: A First Course in Abstract Mathematics" by Ethan D. Bloch (Birkhäuser, 2000).

Algebraic topology is essential to compute topological properties of cyberworlds and the followings are good textbooks:

1. "An Introduction to Topology and Homotopy" by Allan J. Sieradski (PWS-Kent Publishing Company, Boston, MA, USA, 1992). This is most recommended.
2. "A User's Guide to Algebraic Topology" by C. T. J. Dodson and Philip E. Parker (Kluwer Academic Publication, Dordrecht, The Netherlands, 1997).
3. "Algebraic Topology" by Allen Hatcher (Cambridge

University Press, Cambridge, UK, 2002); the up-to-date version can be downloaded freely from `http://www.math.cornell.edu/~hatcher` for personal use.

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Figure 2 as in “Reprint: Tosiyasu L. Kunii, ″Cyber Graphics″, Proceedings of the First International Symposium on Cyber Worlds (CW2002), November 6-8 2002 Tokyo, Japan, pp. 3-7, IEEE Computer Society Press, Los Alamitos, California, November 2002”.

Figure 2. An adjunction space of two disjoint topological spaces X and Y, obtained by starting from X and by attaching Y to it via a continuous function f by identifying each point $y \in Y_0 | Y_0 \subseteq Y$ with its image $f(y) \in X$.

Y_0 is the inside of the hat that touches the head, and the part of the head X touched by the hat is $f(Y_0)$. After the hat is worn, thus, $x \sim f(y)\ |\forall y \in Y_0, Y_0. \subseteq Y, x \in X$.