

VALID COMPUTATIONAL SHAPE MODELING: DESIGN AND IMPLEMENTATION

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Current shape models are targeted at visual presentations for display and design. They lack the validity in their shape properties such as topological-, geometrical- and visual- equivalence, and even continuity. Cellular modeling is a new computational modeling that provides a computationally valid shape model. It also provides a foundation to share shapes among varied applications for extensive reuse. The implementation of cellular modeling via cell attachment tables complies with the standard relational data model. Examples are shown to demonstrate the value of cellular modeling in comparison with the existing typical shape models such as wire frame models, boundary models and solid models. Design and implementation of the cellular modeling examples using cell attachment instance tables are presented.

Keywords: Valid shape modeling, cellular spatial structures, shape equivalence, continuity, cellular modeling design, cellular modeling implementation, cell attachment instance tables, cellular modeling as a standard shape database model.

1. Introduction

Brief statement of the problems confronted us in '70's. There were a number of problems we met when we built raster graphics in late '60s with a frame buffer, as outlined in our paper presented at the first SIGGRAPH conference in 1974 [1]. The most fundamental problems included those of computational shape modeling: valid and invalid. Invalid computational shape modeling means it can compute shapes for raster display that look like fine but are ill defined such that essential shape properties as shape equivalence, shape invariants and continuity are lost. For designing and manufacturing purposes, and also for shape information reusability, such invalidity obviously poses serious troubles and defects. After almost 30 years, the problems have not been solved, or even have not been recognized in research community as serious problems. The reason is simply explained from a modeling point of view. The existing shape modeling usually starts from geometry. There are rich and excellent research results on geometrical shape modeling. If we turn our eyes to the topological level of shape modeling that should be inherited by geometrical shape modeling, very limited research have been conducted. At their bests, what have been meant by topology have been almost within the domain of graph theory. Researchers often discuss the Euler characteristics as graph theoretical invariants, discovered by Euler around two and half centuries ago, but not much more.

Let us look at simplest two cases.

Display assumptions. Here, we set important assumptions for computer display. A point and a line have no size properties, and cannot be displayed on a computer graphics screen as they are. In the following, then, we axiomatize a symbolical display method that presents a point and a line as a pixel and a series of pixels respectively to make them

displayable. To preserve shape boundaries, we further axiomatize, we display overlapped shape boundaries separately without arbitrarily taking their unions.

Any science starts from a set of axioms as assumptions, then goes to formulates or design models, and finally ends by realizing the results through implementation. So is this research.

1.1 Wire frame modeling

The first case is one of the most popular shape modeling in computer-aided design. Computing a wire frame model already posed big difficult problems. We tried to compose a wire frame tetrahedron out of a given set of six equivalent wires. At each vertex of a

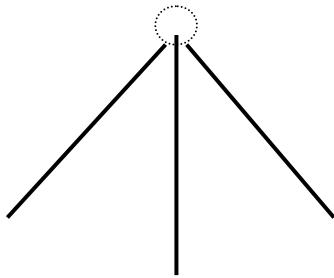


Fig. 1a Three open wires created a vertex lacking a point.

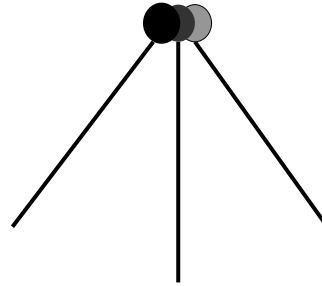


Fig. 1b Three closed wires created a vertex with three points.

tetrahedron, three wires met. When we chose three wires to be open to make them equivalent (Figure 1a), we found the meeting point was missing at the vertex.

Then, as shown in Figure 1b, we took three closed wires to find three points overlapped at the vertex. A “final” solution was proposed to take one closed wire and two open wires (Figure 1c). The picture looked perfect on the display screen. However, the equivalence of wires was lost. A tetrahedron became asymmetric (Figure 2). Depending on the directions of a tetrahedron, we had to process the tetrahedron differently. For example, it

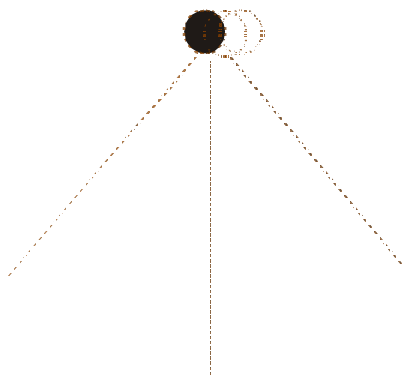
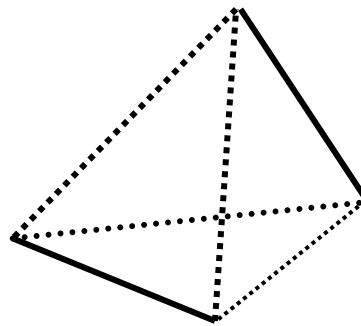


Fig. 1c One closed wire and two open wires looked fine on the screen but they invalidated the equivalence.



— A closed wire
 - - - - - An open wire

Fig. 2 A tetrahedron composed of two closed wires and four open wires, making it asymmetric.

made the identification of a tetrahedron in a scene unnecessarily complex.

1.2 Boundary modeling

Boundary modeling also faced with serious problems. Even such a simple task of composing a boundary model of a tetrahedron out of four equivalent regular triangular planes turned out to be impossible. At each edge where two planes met, open planes yielded no line (Figure 3a), and closed planes gave double lines (Figure 3b). A pair of open and closed

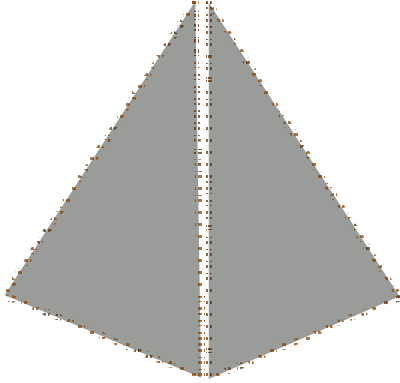


Fig. 3a Two open planes produced an edge lacking a line.

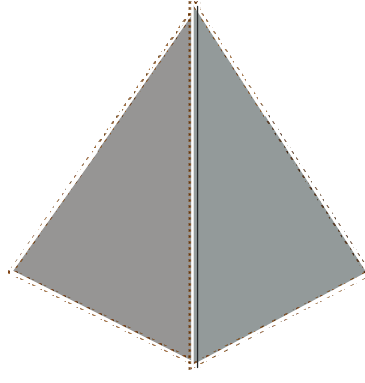


Fig. 3b Two closed planes produced an edge with two lines.

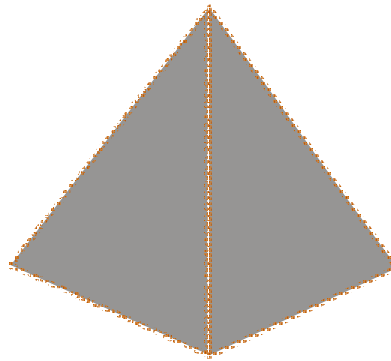


Fig. 3c One closed and one open plans gave an edge with a line, but plane equivalence was lost.

planes gave an edge endowed with a line, but invalidated the plane equivalence (Figure 3c).

Thus, we failed to formulate a valid boundary model.

Even in our shape modeling community, the problems posed to us 30 years ago still remain unsolved. As long as we stay in the theoretical framework of geometric modeling or graph theoretical modeling without considering space continuity and equivalence validity, there is no way to get out of this kind of contradictory problems in modeling shapes.

2. Cellular Structured Spaces

To lay the ground to understand the fundamentals of valid shape modeling, we first look into *cellular structured spaces*, in short *cellular spaces* [2 - 5]. First of all, a *cell* is designed as a topological space X that is topologically equivalent (namely homeomorphic) to an arbitrary dimensional (say n -dimensional) open ball $Int \mathcal{B}^n$, and called an n -cell e^n . We will explain a mapping called *attaching cells* later. As presented by J. H. C. Whitehead in 1950 [6, 7], from X via *attaching cells*, we can *inductively compose* a finite or infinite sequence X^p of cells that are subspaces of X , indexed by integer \mathbb{Z} , namely $\{X^p \mid p \in \mathbb{Z}\}$ called a *filtration* [3], such that

X^p covers X (or, X^p is a *covering* of X),
namely,
 $X = \bigcup_{p \in \mathbb{Z}} X^p$, and
 X^{p-1} is a subspace of X^p ,
namely,
 $X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots \subseteq X^{p-1} \subseteq X^p \subseteq \dots \subseteq X$.

Thus, we can inductively compose a quite general cellular space called a *filtration space* for a cellular space X , as a space X with a filtration composed above, and denote it by $\{X; X^p \mid p \in \mathbb{Z}\}$. We also say that $\{X; X^p \mid p \in \mathbb{Z}\}$ is a *cell decomposition* of a topological space X , or a *partition* of topological space X into subspaces X^p which are cells. Cell composition and decomposition are exploited to compute cellular spatial structures to create valid computational shape modeling.

We can actually build a little bit more structured cellular space, and hence not as general as a filtration space. It is called a closure finite and weak topology space, abbreviated as a *CW-space* constructed by the open subspaces X^p of X . Finite number of cells is enough for our space composition. If we need to think about an infinite case, some extra care is required. Further, as in the most cases in natural sciences as seen in theoretical physics, smoothness, namely the existence of continuous derivatives of all orders, is assumed, and sometimes diffeomorphism, namely differentiability with a differential inverse is further assumed to turn a CW-space into a more special case named a *manifold*. However, in general cases such as shape modeling and applications, usually we cannot have the luxury of enjoying such a limited space built on top of such assumptions.

3. Composing Cellular Structured Spaces via Attaching Cells

We can compose a new cellular structured space Y by attaching an open n -cell e^n to the already composed topological space X , using a *surjective* and *continuous* map f called an *attaching map* (also called an *adjoining map* or an *adjunction map*). In handling geometrical invariants, surjective maps and open maps are of great value to theorize the inherited properties consistently [2]. “A map $f: X \rightarrow Y$ is surjective” means $(\forall y \in Y) (\exists x \in X) [f(x) = y]$. “A map $f: X \rightarrow Y$ is continuous” means “a subset $A \subset Y$ is open in Y if and only if $\{f^{-1}(y) \mid y \in A\}$ is open in X ”. If not continuous, it creates an undesirable

problem such that the boundaries are lost after mapping. The new space Y thus obtained is called an *adjunction space* (or an *adjoining space*). Let us present a more precise definition.

Given two disjoint topological spaces X and Y ,

$$Y \sqcup_f X = Y \sqcup X / \sim$$

is an attaching space (an adjunction space, or an adjoining space) obtained by *attaching* (*gluing*, *adjoining*, or *adjoining*) X to Y by an *attaching map* (*adjunction map*, or an *adjoining map*) f (or by identifying points $x \in X_0 \mid X_0 \subset X$ with their images $f(x) \in Y$, namely by a surjective map f). \sqcup denotes a disjoint union and often a $+$ symbol is used instead (sometimes it is called an “exclusive or”). \sim is an equivalence relation. An *equivalence relation* is simply a relation that is reflexive, symmetric and transitive. It can be a set theoretical equivalence relation, a topological equivalence relation, a geometrical equivalence relation or a homotopic equivalence relation. The transitivity divides the space into a disjoint union of subspaces called *equivalence classes*. The word “classes” is a mal-notation. It means “sets”. The long standing convention can hardly be changed at this stage. If we denote an equivalence class by x / \sim , it is, then,

$$x / \sim = \{ x \in X \mid x \sim y \}.$$

The set of all equivalence classes is denoted as X / \sim , and is called the *quotient space* or the *identification space* of X

$$X / \sim = \{ x / \sim \in 2^X \mid x \in X \} \subseteq 2^X.$$

An attaching map f is a surjective (onto) and continuous map

$$f: X_0 \rightarrow Y,$$

where $X_0 \subset X$.

$X \sqcup Y / \sim$ is a quotient space

$$X \sqcup Y / \sim = X \sqcup Y / (x \sim f(x) \mid \forall x \in X_0) = X \sqcup_f Y.$$

Here is a special case for later use in composing valid shape modeling. Let S^{n-1} be the boundary of a closed n -cell \mathcal{B}^n , namely $\partial\mathcal{B}^n$. That is,

$$\partial\mathcal{B}^n = \mathcal{B}^n - \text{Int}\mathcal{B}^n = \mathcal{B}^n - e^n.$$

Let an attaching map f be a surjective (onto) and continuous map

$$f: S^{n-1} \rightarrow X,$$

namely

$$f: \partial\mathcal{B}^n \rightarrow X.$$

An adjunction space Y is defined as a quotient space

$$Y = X \sqcup_f \mathcal{B}^n = X \sqcup \mathcal{B}^n / \{f(u) \sim u \mid u \in S^{n-1}\}.$$

Given two homotopic maps f and g

$$f, g: S^{n-1} \rightarrow X,$$

namely

$$f, g: \partial\mathcal{B}^n \rightarrow X.$$

then $X \sqcup_f \mathcal{B}^n$ and $X \sqcup_g \mathcal{B}^n$ have the same *homotopy type* (or, are *homotopically equivalent*)

$$X \sqcup_f \mathcal{B}^n \simeq X \sqcup_g \mathcal{B}^n.$$

4. Designing Valid Wire Frame Modeling, Boundary Modeling and Solid Modeling via Attaching Cells – A Case Study of a Tetrahedron Composition-

Now let us design various valid computational shape modeling based on cellular structured

space modeling. We work out an example of simplest cases to compose higher dimensional cellular spaces inductively starting from 0-cells. Since, we looked at the case of a tetrahedron at the beginning, let us go back to it as discussed previously to use it as the design specification of various shape models of a tetrahedron based on cell attachment [8].

4.1 Designing wire frame modeling of a tetrahedron

We can actually design to produce a valid wire frame model X^1 by composing X^1 , starting from X^0 that consists of 0-cells as the elements, via attaching 1-cells to X^0 .

In case of a tetrahedron, X^0 is a set of four vertex points
 $X^0 = \{ e^0_1, e^0_2, e^0_3, e^0_4 \}$.

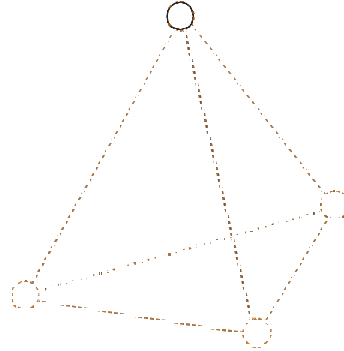


Fig. 4 Composing a wire frame model X^1 of a tetrahedron by attaching six 1-cells e^1_i to X^0 that consists of four 0-cells e^0_i .

A tetrahedron has six edges, and we attach their disjoint union

$$\sqcup_i e^1_i = e^1_1 \sqcup e^1_2 \sqcup e^1_3 \sqcup e^1_4 \sqcup e^1_5 \sqcup e^1_6$$

to X^0 via an attaching map F by identifying each end point $x \in \partial \mathcal{B}^1_i$ of an edge \mathcal{B}^1_i with a vertex point $F(x)$. Then, as shown in Figure 4, we obtain a valid wire frame model X^1 of a tetrahedron such that

$$X^1 = X^0 \sqcup_F (\sqcup_i e^1_i)$$

where $i=1, \dots, 6$, and an attaching map F is

$$F: \sqcup_i \partial \mathcal{B}^1_i \rightarrow X^0.$$

4.2 Designing boundary modeling of a tetrahedron

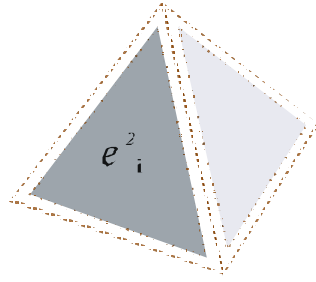


Fig. 5 Composing a boundary model X^2 of a tetrahedron by attaching four 2-cells e^2_i to the wire frame model X^1 .

A valid boundary model X^2 of a tetrahedron is composed starting from the wire frame model X^1 by attaching a disjoint union of four regular triangular planes

$$\sqcup_i e^2_i = e^2_1 \sqcup e^2_2 \sqcup e^2_3 \sqcup e^2_4$$

to it via an attaching map G

$$G: \sqcup_i \partial \mathcal{B}^2_i \rightarrow X^1 \text{ where } i = 1, \dots, 4$$

by identifying the boundary line $\lambda \in \partial \mathcal{B}^2_i$ of each plane e^2_i with a part $G(\lambda)$ of the wire frame model X^1 . The boundary model X^2 thus composed is, as shown in Figure 5,

$$X^2 = X^1 \sqcup_G (\sqcup_i e^2_i)$$

where $i = 1, \dots, 4$.

4.3 Designing solid modeling of a tetrahedron

Now we design a valid solid model X^3 of a tetrahedron starting from the boundary model X^2 by attaching a 3-dimensional ball \mathcal{B}^3 to it via an attaching map H

$$H: \partial \mathcal{B}^3 \rightarrow X^2$$

by identifying the boundary plane $\pi \in \partial \mathcal{B}^3$ of the ball \mathcal{B}^3 with the boundary plane $H(\pi)$ of the boundary model X^2 . The solid model X^3 obtained is (Figure 6)

$$X^3 = X^2 \sqcup_H e^3.$$

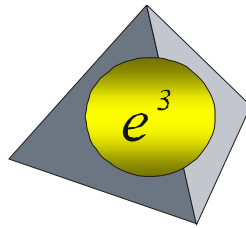


Fig. 6 Composing a solid model X^3 of a tetrahedron by attaching a 3-cell e^3 to the boundary model X^2 .

5. Implementation

1. Implementing cellular modeling is fairly straightforward. The implementation of tetrahedron composition via cell attachment is realized as a set of *cell attachment instance tables*. The *cell attachment instance tables* $T(e_i^n)$ implements the design of an n-dimensional space X^n . As already explained X^n is designed by attaching n-dimensional closed cells e_i^n to the already constructed (n-1)-dimensional space X^{n-1} . $T(e_i^n)$ is the subsets of the Cartesian product of a \mathcal{B}_i^n cell boundary $\partial\mathcal{B}_i^n$ and X^{n-1} such that:

$$T(e_i^n) \subseteq \partial\mathcal{B}_i^n \times X^{n-1}.$$

For $i = 1, 2, \dots, k$, the number of cell attachment instance tables is k . Our implementation based on cell attachment instance tables stands for the realization of low level representation of cell attachment as a continuous and surjective identification (quotient) mapping. It complies with a relational data model as a standard in data base management. This means that our implementation turns cellular modeling into a standard shape database model. Actually, cellular model itself stands for a further versatile data model we named a cellular data model [10, 11].

5.1 Implementing wire frame modeling

For each edge e_i^1 ($i = 1, \dots, 6$) is an open 1-cell, there are two end points $x_1, x_2 \in \partial\mathcal{B}_i^1$, each of which being identified with a vertex point e_j^0 ($j = 1, \dots, 4$) to create a row in a cell attachment instance table $T(e_i^1)$ for the edge e_i^1 . For the two end points, the table has two rows. Corresponding to six edges e_i^1 ($i = 1, \dots, 6$), we obtain six cell attachment instance tables as shown in Table 1 implementing the wire frame model X^1 of a tetrahedron.

$$T(e_i^1)$$

$\partial\mathcal{B}_i^1$	X^0
x_1	e_j^0
x_2	e_k^0

5.2 Implementing boundary modeling

The boundary model X^2 is implemented starting from the implemented wire frame model X^1 . For four closed triangular planes e^2_j ($j = 1, \dots, 4$), we create four cell attachment instance

Table 2. four cell attachment instance tables $T(e^2_i)$, $i = 1, \dots, 4$, implementing the boundary model X^2 of a tetrahedron.

$$T(e^2_i)$$

$\partial\mathcal{B}_i^2$	X^1
λ_1	e^1_j
λ_2	e^1_k
λ_3	e^1_l

$\{i, j, k\} \subset \{1, \dots, 6\}$, and
 $e^1_i \in \{e^1_1, \dots, e^1_6\}$.

tables, one for each closed triangular plane. Each table $T(e^2_i)$ has three rows and each row identifies one edge out of three edges $\lambda_1, \lambda_2, \lambda_3 \in \partial\mathcal{B}_i^2$ of a given triangular plane e^2_i with the part e^1_i of the wire frame model X^1 where $i = 1, \dots, 3$. e^1_i is an open 1-cell such that $e^1_i \in \{e^1_1, \dots, e^1_6\}$. Four cell attachment tables $T(e^2_i)$, $i = 1, \dots, 4$, implement the boundary model X^2 of a tetrahedron as shown in Table 2.

5.3 Implementing solid modeling

Finally we implement the solid modeling X^3 of a tetrahedron based on the implemented boundary model X^2 . It is just one cell attachment to identify the boundary plane $\pi = \partial\mathcal{B}^3$ of an open ball e^3 with the boundary model X^2 . The cell attachment instance table $T(e^3)$ has only one row as shown in Table 3.

Table 3. A cell attachment instance table $T(e^3)$, implementing the solid model X^3 of a tetrahedron.

$$T(e^3)$$

$\partial\mathcal{B}^3$	X^2
π	X^2

Here, we make a short excursion in this new field of cellular modeling design and implementation. If we like to implement a different design such that a part $\pi_i \subseteq \pi$ of the surface $\partial\mathcal{B}^3$ of an open ball e^3 is attached to each triangular plane e^2_i of X^2 , it is implemented as follows:

A part π_i of the surface $\partial\mathcal{B}^3$ of an open ball e^3 placed inside the boundary model X^2 and facing a triangular plane e^2_i of the boundary model X^2 is identified, to create a row in a cell

Table 4. A cell attachment instance table $T(e^3)$, implementing the solid model X^3 of a tetrahedron via piece-wise cell attachment to triangular planes.

$T(e^3)$

$\partial\mathcal{B}^3$	X^2
π_1	e_1^2
π_2	e_2^2
π_3	e_3^2
π_4	e_4^2

attachment instance table $T(e^3)$. Let us perform the identification for $i=1, \dots, 4$ to create four rows in $T(e^3)$. In this case, the table $T(e^3)$ as shown in Table 4 serves as the implementation of the solid modeling X^3 of a tetrahedron.

5.4 Remarks on implementation

We can go further to implement the models at the level of geometry inheriting the implementation of cellular modeling that stands for implementing algebraic topological properties of shapes. By implementing visualization models inheriting the geometrical level implementation, we can display the shapes on computer graphics screen. Our implementation above has laid the ground for such implementation as the foundation.

6. Discussions and Conclusions

Basic research is a lonely job. Although I have used “we” so far, it has been “I”. I have been wondering at the problems we were confronted 30 years ago since then, alone. A fundamental and hence basic research requires a lifelong lonely mental journey any way. *Valid shape modeling* has solved the problems postulated at the beginning, as shown so far. The type of shape equivalence presented here is preserved at all the levels of shape representations: homotopical, topological, geometrical and visualization levels. It has the following additional advantages:

1. *Valid shape modelings* turns shape components into reusable resources as cells and store them in shape databases that conform with the standard data modeling of the relational model so that we can reassemble them to create new shapes freely. Valid shape modeling also guarantees the shape databases to be valid;
2. Communication of shape information enjoys high compression ratios by sending only the cellular component identifiers and cellular attachment operation identifiers as needed. This is shown in the implementation as cell attachment instance tables.

Important findings are related to a pair of cellular spatial algorithms and cellular spatial structures:

Cell decomposition and composition algorithms and cellular spatial structures.

A more comprehensive theory has been completed and is to appear elsewhere. Far wider and deeper applications are studied currently such as the visualizations of financial trading including financial structural changes and M&A structures, and of flexible shapes. It is actually the case that cyberworlds built as worlds of information need to be composed as valid worlds. The composition is achieved based on the theoretical framework presented in this research. High demands are expected to come from emerging major social activities such as electronic financing, electronic commerce, media intensive political campaigns, and complex social system designs and manufacturing. They are all shifting into information worlds often called cyberworlds [9].

7. Acknowledgments

A comment of an anonymous reviewer was valuable to revise “*Display axioms*” at the beginning of the paper, resulting in adding the last sentence there to make the later discussion on Figure 1b clearer.

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[11] Numbers of cellular information model papers are listed on the home page:
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